Last time: Boundary Value Problems, PDE classification
Today: Numerical solution of hyperbolic PDEs
We now consider how to solve $u_t + cu_x = 0$ equation using a finite difference method.

**Question**: Why finite differences? Why not just use characteristics?

**Answer**: Characteristics actually are a viable option for computational methods, and are used in practice. However, **characteristic methods** can become very complicated in 2D or 3D, or for nonlinear problems. Finite differences are a much more practical choice in most circumstances.
Hyperbolic PDEs: Numerical Approximation

Advection equation is an Initial Boundary Value Problem (IBVP)

We impose an initial condition, and a boundary condition (only one BC since first order PDE)

A finite difference approximation leads to a grid in the $xt$-plane
The first step in developing a finite difference approximation for the advection equation is to consider the **CFL condition**\(^1\)

The CFL condition is a **necessary condition** for the convergence of a finite difference approximation of a hyperbolic problem.

Suppose we discretize \( u_t + cu_x = 0 \) in space and time using the explicit (in time) scheme

\[
\frac{U_j^{n+1} - U_j^n}{\Delta t} + c \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0
\]

Here \( U_j^n \approx u(t_n, x_j) \), where \( t_n = n\Delta t, x_j = j\Delta x \)

\(^1\)Courant–Friedrichs–Lewy condition, published in 1928
This can be rewritten as

\[ U_j^{n+1} = U_j^n - \frac{c\Delta t}{\Delta x} (U_j^n - U_{j-1}^n) = (1 - \nu)U_j^n + \nu U_{j-1}^n \]

where

\[ \nu \equiv \frac{c\Delta t}{\Delta x} \]

We can see that \( U_j^{n+1} \) depends only on \( U_j^n \) and \( U_{j-1}^n \)
Definition: Domain of dependence of $U_j^{n+1}$ is the set of values that $U_j^{n+1}$ depends on.
The domain of dependence of the exact solution $u(t_{n+1}, x_j)$ is determined by the characteristic curve passing through $(t_{n+1}, x_j)$.

CFL Condition:

For a convergent scheme, the domain of dependence of the PDE must lie within the domain of dependence of the numerical method.
Suppose the dashed line indicates characteristic passing through \((t_{n+1}, x_j)\), then the scheme below satisfies the CFL condition.
The scheme below does not satisfy the CFL condition
The scheme below does not satisfy the CFL condition (here $c < 0$)
Question: What goes wrong if the CFL condition is violated?
Answer: The exact solution $u(x, t)$ depends on initial value $u_0(x_0)$, which is outside the numerical method’s domain of dependence.

Therefore, the numerical approx. to $u(x, t)$ is “insensitive” to the value $u_0(x_0)$, which means that the method cannot be convergent.
Hyperbolic PDEs: Numerical Approximation

Note that CFL is only a necessary condition for convergence

Its great value is its simplicity: CFL allows us to easily reject F.D. schemes for hyperbolic problems with very little investigation

For example, for $u_t + cu_x = 0$, the scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + c\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0 \quad (\star)$$

cannot be convergent if $c < 0$

**Question**: What small change to $(\star)$ would give a better method when $c < 0$?
If \( c > 0 \), then we require \( \nu \equiv \frac{c\Delta t}{\Delta x} \leq 1 \) in (*) for CFL to be satisfied.
Hyperbolic PDEs: Upwind method

As foreshadowed earlier, we should pick our method to reflect the direction of propagation of information.

This motivates the upwind scheme for \( u_t + cu_x = 0 \)

\[
U_j^{n+1} = \begin{cases} 
U_j^n - c \frac{\Delta t}{\Delta x} (U_j^n - U_{j-1}^n), & \text{if } c > 0 \\
U_j^n - c \frac{\Delta t}{\Delta x} (U_{j+1}^n - U_j^n), & \text{if } c < 0 
\end{cases}
\]

The upwind scheme satisfies CFL condition if \( |\nu| \equiv |c \Delta t / \Delta x| \leq 1 \)

\( \nu \) is often called the CFL number.
Hyperbolic PDEs: Central difference method

Another method that seems appealing is the central difference method:

\[
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + c \frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\Delta x} = 0
\]

This satisfies CFL for \(|\nu| \equiv |c\Delta t/\Delta x| \leq 1\), regardless of sign\((c)\)

We shall see shortly, however, that this is a bad method!
Recall that truncation error is “what is left over when we substitute exact solution into the numerical approximation”

Truncation error is analogous for PDEs, e.g. for the \((c > 0)\) upwind method, truncation error is:

\[
T_j^n = \frac{u(t^{n+1}, x_j) - u(t^n, x_j)}{\Delta t} + c \frac{u(t^n, x_j) - u(t^n, x_{j-1})}{\Delta x}
\]

The order of accuracy is then the largest \(p\) such that

\[
T_j^n = O((\Delta x)^p + (\Delta t)^p)
\]
Hyperbolic PDEs: Accuracy

See Lecture: For the upwind method, we have

\[ T_j^n = \frac{1}{2} \left[ \Delta t u_{tt}(t^n, x_j) - c \Delta x u_{xx}(t^n, x_j) \right] + \text{H.O.T.} \]

Hence the upwind scheme is first order accurate
Hyperbolic PDEs: Accuracy

Just like with ODEs, truncation error is related to convergence in the limit $\Delta t, \Delta x \to 0$

Note that to let $\Delta t, \Delta x \to 0$, we generally need to decide on a relationship between $\Delta t$ and $\Delta x$

e.g. to let $\Delta t, \Delta x \to 0$ for the upwind scheme, we would set $\frac{c\Delta t}{\Delta x} = \nu \in (0, 1]$; this ensures CFL is satisfied for all $\Delta x, \Delta t$
In general, convergence of a finite difference method for a PDE is related to both its truncation error and its stability.

We’ll discuss this in more detail shortly, but first we consider how to analyze stability via Fourier stability analysis.
Let’s suppose that $U_j^n$ is periodic on the grid $x_1, x_2, \ldots, x_n$. 
Hyperbolic PDEs: Stability

Then we can represent $U^m_j$ as a linear combination of sin and cos functions, i.e. Fourier modes.

Or, equivalently, as a linear combination of complex exponentials, since $e^{ikx} = \cos(kx) + i\sin(kx)$ so that

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}), \quad \cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$$
For simplicity, let’s just focus on only one of the Fourier modes.

In particular, we consider the ansatz $U_j^n(k) \equiv \lambda(k)^n e^{ikx_j}$, where $k$ is the wave number and $\lambda(k) \in \mathbb{C}$.

**Key idea**: Suppose that $U_j^n(k)$ satisfies our finite difference equation, then this will allow us to solve for $\lambda(k)$.

The value of $|\lambda(k)|$ indicates whether the Fourier mode $e^{ikx_j}$ is amplified or damped.

If $|\lambda(k)| \leq 1$ for all $k$ then the scheme does not amplify any Fourier modes $\implies$ stable!

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\[2\] In general a solution for $\lambda(k)$ exists, which justifies our choice of ansatz.
Hyperbolic PDEs: Stability

We now perform Fourier stability analysis for the \((c > 0)\) upwind scheme (recall that \(\nu = \frac{c \Delta t}{\Delta x}\)):

\[
U_j^{n+1} = U_j^n - \nu(U_j^n - U_{j-1}^n)
\]

Substituting in \(U_j^n(k) = \lambda(k)^n e^{ik(j\Delta x)}\) gives

\[
\lambda(k)e^{ik(j\Delta x)} = e^{ik(j\Delta x)} - \nu(e^{ik(j\Delta x)} - e^{ik((j-1)\Delta x)})
\]

\[
= e^{ik(j\Delta x)} - \nu e^{ik(j\Delta x)}(1 - e^{-ik\Delta x})
\]

Hence

\[
\lambda(k) = 1 - \nu(1 - e^{-ik\Delta x}) = 1 - \nu(1 - \cos(k\Delta x) + i \sin(k\Delta x))
\]
Hyperbolic PDEs: Stability

It follows that

\[ |\lambda(k)|^2 = [(1 - \nu) + \nu \cos(k\Delta x)]^2 + [\nu \sin(k\Delta x)]^2 \]
\[ = (1 - \nu)^2 + \nu^2 + 2\nu(1 - \nu) \cos(k\Delta x) \]
\[ = 1 - 2\nu(1 - \nu)(1 - \cos(k\Delta x)) \]

and from the trig. identity \((1 - \cos(\theta)) = 2\sin^2\left(\frac{\theta}{2}\right)\), we have

\[ |\lambda(k)|^2 = 1 - 4\nu(1 - \nu) \sin^2 \left(\frac{1}{2}k\Delta x\right) \]

Due to the CFL condition, we first suppose that \(0 \leq \nu \leq 1\)

It then follows that \(0 \leq 4\nu(1 - \nu) \sin^2 \left(\frac{1}{2}k\Delta x\right) \leq 1\), and hence

\[ |\lambda(k)| \leq 1 \]
Hyperbolic PDEs: Stability

In contrast, consider stability of the central difference approx.:

\[
\frac{U_j^{n+1} - U_j^n}{\Delta t} + c \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0
\]

Recall that this also satisfies the CFL condition as long as \(|\nu| \leq 1\)

But Fourier stability analysis yields

\[
\lambda(k) = 1 - \nu i \sin(k\Delta x) \implies |\lambda(k)|^2 = 1 + \nu^2 \sin^2(k\Delta x)
\]

and hence \(|\lambda(k)| > 1\) (unless \(\sin(k\Delta x) = 0\)), i.e. unstable!
Consistency

We say that a numerical scheme is consistent with a PDE if its truncation error tends to zero as $\Delta x, \Delta t \to 0$

For example, any first (or higher) order scheme is consistent
Then a fundamental theorem in Scientific Computing is the Lax\textsuperscript{3} Equivalence Theorem:

For a consistent finite difference approx. to a linear evolutionary problem, the stability of the scheme is necessary and sufficient for convergence.

This theorem refers to linear evolutionary problems, e.g. linear hyperbolic or parabolic PDEs.

\textsuperscript{3}Peter Lax, Courant Institute, NYU
Lax Equivalence Theorem

We know how to check consistency: Derive the truncation error.

We know how to check stability: Fourier stability analysis.

Hence, from Lax, we have a general approach for verifying convergence.

Also, as with ODEs, convergence rate is determined by truncation error.
Lax Equivalence Theorem

Note that strictly speaking Fourier stability analysis only applies for periodic problems.

However, it can be shown that conclusions of Fourier stability analysis hold true more generally.

Hence Fourier stability analysis is the standard tool for examining stability of finite difference methods for PDEs.
So far, we have developed full discretizations (both space and time) of the advection equation, and considered accuracy and stability.

However, it can be helpful to consider semi-discretizations, where we discretize only in space, or only in time.

For example, discretizing $u_t + c(t, x)u_x = 0$ in space\(^4\) using a backward difference formula gives

$$\frac{\partial U_j(t)}{\partial t} + c_j(t)\frac{U_j(t) - U_{j-1}(t)}{\Delta x} = 0, \quad j = 1, \ldots, n$$

\(^4\)Here we show an example where $c$ is not constant.
Hyperbolic PDEs: Semi-discretization

This gives a system of ODEs, $U_t = f(t, U(t))$, where $U(t) \in \mathbb{R}^n$ and

$$f(t, U(t)) \equiv -c_j(t) \frac{U_j(t) - U_{j-1}(t)}{\Delta x}$$

We could approximate this ODE using forward Euler (to get our Upwind scheme):

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = f(t^n, U^n) = -c_j^n \frac{U_j^n - U_{j-1}^n}{\Delta x}$$

Or backward Euler:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = f(t^{n+1}, U^{n+1}) = -c_j^{n+1} \frac{U_j^{n+1} - U_{j-1}^{n+1}}{\Delta x}$$
Hyperbolic PDEs: Method of Lines

Or we could use a “black box” ODE solver, such as ode45, to solve the system of ODEs

This “black box” approach is called the method of lines

The name “lines” is because we solve each $U_j(t)$ for a fixed $x_j$, i.e. a line in the $xt$-plane

With method of lines we let the ODE solver to choose step sizes $\Delta t$ to obtain a stable and accurate scheme
We now briefly return to the wave equation:

\[ u_{tt} - c^2 u_{xx} = 0 \]

In one spatial dimension, this models, say, vibrations in a taut string.
The Wave Equation

Many schemes have been proposed for the wave equation

One good option is to use central difference approximations\(^5\) for both \(u_{tt}\) and \(u_{xx}\):

\[
\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\Delta t^2} - c^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0
\]

Key points:
- Truncation error analysis \(\implies\) second-order accurate
- Fourier stability analysis \(\implies\) stable for \(0 \leq c\Delta t / \Delta x \leq 1\)
- Two-step method in time, need a one-step method to "get started"

\(^5\)Can arrive at the same result by discretizing the equivalent first order system