Last time: Singular Value Decomposition
Today: Low-rank approximation, numerical calculus
Low-Rank Approximation via the SVD

One of the most useful properties of the SVD is that it allows us to obtain an optimal low-rank approximation to \( A \)

We can recast SVD as

\[
A = \sum_{j=1}^{r} \sigma_j u_j v_j^T
\]

Follows from writing \( \Sigma \) as sum of \( r \) matrices, \( \Sigma_j \), where

\[
\Sigma_j \equiv \text{diag}(0, \ldots, 0, \sigma_j, 0, \ldots, 0)
\]

Each \( u_j v_j^T \) is a rank one matrix: each column is a scaled version of \( u_j \)
Low-Rank Approximation via the SVD

**Theorem:** For any $0 \leq \nu \leq r$, let $A_{\nu} \equiv \sum_{j=1}^{\nu} \sigma_j u_j v_j^T$, then

$$\|A - A_{\nu}\|_2 = \inf_{B \in \mathbb{R}^{m \times n}, \text{rank}(B) \leq \nu} \| A - B \|_2 = \sigma_{\nu+1}$$

That is:

- $A_{\nu}$ gives us the closest rank $\nu$ matrix to $A$, measured in the 2-norm
- The error in $A_{\nu}$ is given by the first *omitted* singular value
Low-Rank Approximation via the SVD

A similar result holds in the Frobenius norm:

\[ \|A - A_\nu\|_F = \inf_{B \in \mathbb{R}^{m \times n}, \text{rank}(B) \leq \nu} \|A - B\|_F = \sqrt{\sigma_{\nu+1}^2 + \cdots + \sigma_r^2} \]
Low-Rank Approximation via the SVD

These theorems indicate that the SVD is an effective way to *compress* data encapsulated by a matrix!

If singular values of $A$ decay rapidly, can approximate $A$ with few rank one matrices (only need to store $\sigma_j, u_j, v_j$ for $j = 1, \ldots, \nu$)

**Example:** Image compression via the SVD
Motivation

Since the time of Newton, calculus has been ubiquitous in science.

Many (most?) calculus problems that arise in applications do not have closed-form solutions.

Numerical approximation is essential!

Epitomizes idea of Scientific Computing as developing and applying numerical algorithms to problems of continuous mathematics.

In this Unit we will consider:

- Numerical integration
- Numerical differentiation
- Numerical methods for ordinary differential equations
- Numerical methods for partial differential equations
Integration
Integration

Approximating a definite integral using a numerical method is called quadrature.

The familiar Riemann sum idea suggests how to perform quadrature.

We will examine more accurate/efficient quadrature methods.
Question: Why is quadrature important?

We know how to evaluate many integrals analytically, e.g.

\[ \int_0^1 e^x \, dx \quad \text{or} \quad \int_0^\pi \cos x \, dx \]

But how about \( \int_1^{2000} \exp(\sin(\cos(\sinh(\cosh(\tan^{-1}(\log(x))))))) \, dx \)?
We can numerically approximate this integral in Python using quadrature.

Python 2.7.5 (default, Mar 9 2014, 22:15:05)
[GCC 4.2.1 Compatible Apple LLVM 5.0 (clang-500.0.68)] on darwin
Type "help", "copyright", "credits" or "license" for more information.

```python
>>> import scipy.integrate as spi
>>> from math import *
>>> def f(x):
...     return exp(sin(cos(sinh(cosh(atan(log(x)))))))
...     return exp(sin(cos(sinh(cosh(atan(log(x)))))))
...     return exp(sin(cos(sinh(cosh(atan(log(x)))))))
...     return exp(sin(cos(sinh(cosh(atan(log(x)))))))
...     return exp(sin(cos(sinh(cosh(atan(log(x)))))))
...     return exp(sin(cos(sinh(cosh(atan(log(x)))))))
>>> spi.quad(f,1,2000)
(1514.7806778270258, 4.231109728875231e-06)
```
Integration

Quadrature also generalizes naturally to higher dimensions, and allows us to compute integrals on irregular domains.

For example, we can approximate an integral on a triangle based on a finite sum of samples at quadrature points.

Three different quadrature rules on a triangle.
Integration

Can then evaluate integrals on complicated regions by “triangulating” (AKA “meshing”)

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Differentiation
Numerical differentiation is another fundamental tool in Scientific Computing

We have already discussed the most common, intuitive approach to numerical differentiation: finite differences

\[
\begin{align*}
    f'(x) &= \frac{f(x + h) - f(x)}{h} + O(h) \quad \text{(forward difference)} \\
    f'(x) &= \frac{f(x) - f(x - h)}{h} + O(h) \quad \text{(backward difference)} \\
    f'(x) &= \frac{f(x + h) - 2f(x) + f(x - h)}{2h} + O(h^2) \quad \text{(centered difference)} \\
    f''(x) &= \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} + O(h^2) \quad \text{(centered, 2nd deriv.)}
\end{align*}
\]
Differentiation

We will see how to derive these and other finite difference formulas and quantify their accuracy.

Wide range of choices, with trade-offs in terms of:

- accuracy
- stability
- complexity
Differentiation

We saw at the start of the course that finite differences can be sensitive to rounding error when $h$ is “too small”

But in most applications we obtain sufficient accuracy with $h$ large enough that rounding error is still negligible\(^1\)

Hence finite differences generally work very well, and provide a very popular approach to solving problems involving derivatives

\(^1\)That is, $h$ is large enough so that rounding error is dominated by discretization error
ODEs
The most common situation in which we need to approximate derivatives is in order to solve differential equations.

Ordinary Differential Equations (ODEs): Differential equations involving functions of one variable.

Some example ODEs:

- \( y'(t) = y^2(t) + t^4 - 6t, \ y(0) = y_0 \) is a \textbf{first order} Initial Value Problem (IVP) ODE.
- \( y''(x) + 2xy(x) = 1, \ y(0) = y(1) = 0 \) is a \textbf{second order} Boundary Value Problem (BVP) ODE.
ODEs: IVP

A familiar IVP ODE is Newton’s Second Law of Motion: suppose position of a particle at time $t \geq 0$ is $y(t) \in \mathbb{R}$

$$y''(t) = \frac{F(t, y, y')}{m}, \quad y(0) = y_0, y'(0) = v_0$$

This is a scalar ODE ($y(t) \in \mathbb{R}$), but it’s common to simulate a system of $N$ interacting particles

e.g. $F$ could be gravitational force due to other particles, then force on particle $i$ depends on positions of the other particles
ODEs: IVP

$N$-body problems are the basis of many cosmological simulations:
Recall galaxy formation simulations from Unit 0

Computationally expensive when $N$ is large!
ODEs: BVP

ODE boundary value problems are also important in many circumstances.

For example, steady state heat distribution in a “1D rod”

Apply heat source $f(x) = x^2$, impose “zero” temperature at $x = 0$, insulate at $x = 1$:

$$-u''(x) = x^2, \quad u(0) = 0, u'(1) = 0$$
We can approximate via finite differences: use F.D. formula for $u''(x)$.
PDEs
PDEs

It is also natural to introduce time-dependence for the temperature in the “1D rod” from above.

Hence now $u$ is a function of $x$ and $t$, so derivatives of $u$ are partial derivatives, and we obtain a partial differential equation (PDE).

For example, the time-dependent heat equation for the 1D rod is given by:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = x^2, \quad u(x, 0) = 0, u(0, t) = 0, \frac{\partial u}{\partial x}(1, t) = 0$$

This is an Initial-Boundary Value Problem (IBVP)
Also, when we are modeling continua\textsuperscript{2} we generally also need to be able to handle 2D and 3D domains

e.g. 3D analogue of time-dependent heat equation on a domain \( \Omega \subset \mathbb{R}^3 \) is

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = f(x, y, z), \quad u = 0 \text{ on } \partial \Omega
\]

\textsuperscript{2}e.g. temperature distribution, fluid velocity, electromagnetic fields,...
This equation is typically written as

\[
\frac{\partial u}{\partial t} - \Delta u = f(x, y, z), \quad u = 0 \text{ on } \partial \Omega
\]

where \( \Delta u \equiv \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \)

Here we have:

- The Laplacian, \( \Delta \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \)

- The gradient, \( \nabla \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \)
PDEs

Can add a “transport” term to the heat equation to obtain the convection-diffusion equation, e.g. in 2D we have

\[ \frac{\partial u}{\partial t} + (w_1(x, y), w_2(x, y)) \cdot \nabla u - \Delta u = f(x, y), \quad u = 0 \text{ on } \partial \Omega \]

\( u(x, t) \) models concentration of some substance, e.g. pollution in a river with current \((w_1, w_2)\)
PDEs

Numerical methods for PDEs are a major topic in scientific computing

Recall examples from Unit 0:

The finite difference method is an effective approach for a wide range of problems, hence we focus on F.D. in AM205³

³There are many important alternatives, e.g. finite element method, finite volume method, spectral methods, boundary element methods...
Numerical calculus encompasses a wide range of important topics in scientific computing!

As always, we will pay attention to stability, accuracy and efficiency of the algorithms that we consider.
Suppose we want to evaluate the integral \( I(f) \equiv \int_{a}^{b} f(x)dx \)

We can proceed as follows:

1. Approximate \( f \) using a polynomial interpolant \( p_n \)
2. Evaluate \( Q_n(f) \equiv \int_{a}^{b} p_n(x)dx \), since we know how to integrate polynomials

\( Q_n(f) \) provides a quadrature formula, and we should have \( Q_n(f) \approx I(f) \)

A quadrature rule based on an interpolant \( p_n \) at \( n + 1 \) equally spaced points in \([a, b]\) is known as Newton–Cotes formula of order \( n \)
Newton–Cotes Quadrature

Let $x_k = a + kh$, $k = 0, 1, \ldots, n$, where $h = (b - a)/n$

We write the interpolant of $f$ in Lagrange form as

$$p_n(x) = \sum_{k=0}^{n} f(x_k)L_k(x), \quad \text{where} \quad L_k(x) \equiv \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$$

Then

$$Q_n(f) = \int_{a}^{b} p_n(x)dx = \sum_{k=0}^{n} f(x_k) \int_{a}^{b} L_k(x)dx = \sum_{k=0}^{n} w_k f(x_k)$$

where $w_k \equiv \int_{a}^{b} L_k(x)dx \in \mathbb{R}$ is the $k$th quadrature weight
Newton–Cotes Quadrature

Note that quadrature weights do not depend on \( f \), hence can be precomputed and stored.

\( n = 1 \implies \) Trapezoid rule (See lecture)

\[
Q_2(f) = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \quad \text{Simpson rule}
\]

We can also develop higher-order Newton–Cotes formulae in the same way.
Error Estimates

Let \( E_n(f) \equiv I(f) - Q_n(f) \)

Then

\[
E_n(f) = \int_a^b f(x)dx - \sum_{k=0}^n w_k f(x_k)
\]

\[
= \int_a^b f(x)dx - \sum_{k=0}^n \left( \int_a^b L_k(x)dx \right) f(x_k)
\]

\[
= \int_a^b f(x)dx - \int_a^b \left( \sum_{k=0}^n L_k(x) f(x_k) \right) dx
\]

\[
= \int_a^b f(x)dx - \int_a^b p_n(x)dx
\]

\[
= \int_a^b (f(x) - p_n(x)) dx
\]

And we have an expression for \( f(x) - p_n(x) \)
Error Estimates

Recall from I.2

\[ f(x) - p_n(x) = \frac{f^{n+1}(\theta)}{(n+1)!} (x - x_0) \cdots (x - x_n) \]

Hence

\[ |E_n(f)| \leq \frac{M_{n+1}}{(n+1)!} \int_a^b |(x - x_0)(x - x_1) \cdots (x - x_n)| \, dx \]

where \( M_{n+1} = \max_{\theta \in [a,b]} |f^{n+1}(\theta)| \)
Error Estimates

See lecture: Trapezoid rule error bound

\[ |E_1(f)| \leq \frac{(b - a)^3}{12} M_2 \]

The bound for \( E_n \) depends directly on the integrand \( f \) (via \( M_{n+1} \))

Just like with the Lebesgue constant, it is informative to be able to compare quadrature rules independently of the integrand
Theorem: If $Q_n$ integrates polynomials of degree $n$ exactly, then $\exists C_n > 0$ such that $|E_n(f)| \leq C_n \min_{p \in \mathbb{P}_n} \|f - p\|_\infty$

Proof: For $p \in \mathbb{P}_n$, we have

$$|I(f) - Q_n(f)| \leq |I(f) - I(p)| + |I(p) - Q_n(f)|$$
$$= |I(f - p)| + |Q_n(f - p)|$$
$$\leq \int_a^b dx \|f - p\|_\infty + \left( \sum_{k=0}^n |w_k| \right) \|f - p\|_\infty$$
$$\equiv C_n \|f - p\|_\infty$$

where

$$C_n \equiv b - a + \sum_{k=0}^n |w_k|$$