AM205: Take-home midterm exam (Fall 2018)

This exam was posted at 5 PM on November 8th. Answers are due at 5 PM on November 10th. Solutions should be uploaded to Canvas.

For queries, contact the teaching staff using a private message on Piazza—do not post questions publicly. Any clarifications will be posted on Piazza.

The exam is open book—any class notes, books, or online resources can be used. The exam must be completed by yourself and no collaboration with classmates or others is allowed. The exam will be graded out of forty points. Point values for each question are given in square brackets.

1. A stencil based on least squares [8].

(a) Consider a smooth function \( f(x) \) and let \( f_k = f(x_k) \) where \( x_k = kh \). Calculate\(^2\) the coefficients \( a_{-1}, a_0, a_1, a_2, a_3 \) to make a fourth-order accurate finite-difference formula,

\[
 f'_\text{diff}_1(x_k) = \frac{a_{-1} f_{k-1} + a_0 f_k + a_1 f_{k+1} + a_2 f_{k+2} + a_3 f_{k+3}}{h}. \tag{1}
\]

As discussed in the lecture notes, one way to do this is by interpolating a quartic through the five given points \((x_j, f_j)\), and then calculating the derivative of this quartic at \( x_k \). In this question we will compare this standard stencil with a variation.

(b) Suppose that instead of fitting a quartic through the five points \((x_j, f_j)\), we instead find the least squares best-fit quadratic through them. By differentiating the quadratic and evaluating it at \( x_k \), find a numerical approximation to the derivative,

\[
 f'_\text{diff}_2(x_k) = \frac{b_{-1} f_{k-1} + b_0 f_k + b_1 f_{k+1} + b_2 f_{k+2} + b_3 f_{k+3}}{h}, \tag{2}
\]

for some \( b_{-1}, b_0, b_1, b_2, \) and \( b_3 \).\(^3\)

(c) Consider the function \( f(x) = \cos(\exp(3 \cos x)) \) on the periodic interval \([0, 2\pi]\), and introduce a grid \( x_k = kh \) where \( k = 0, \ldots, N - 1 \) and \( h = 2\pi / N \). For \( N = 1024 \), calculate the root mean squared error of the two methods,\(^4\)

\[
 E_1 = \sqrt{\frac{1}{N} \sum_{k=0}^{N-1} (f'(x_k) - f'_{\text{diff}_1}(x_k))^2 }, \quad E_2 = \sqrt{\frac{1}{N} \sum_{k=0}^{N-1} (f'(x_k) - f'_{\text{diff}_2}(x_k))^2 }. \tag{3}
\]

Here \( f'(x_k) \) is the exact derivative of \( f \) at \( x_k \). Which method gives a more accurate result?

(d) Let \( g_k = f_k + 10^{-4} z_k \) where the \( z_k \) are independently normally distributed random numbers with mean 0 and standard deviation 1. Calculate new values of \( E_1 \) and \( E_2 \), where each value of \( f_k \) in Eqs. 1 & 2 is replaced with \( g_k \). Which method gives a more accurate result? Compare your results to part (c).

\(^1\)You may use numerical linear algebra routines in any part of this question.
\(^2\)You may use any method to do this, e.g. Taylor expansions or Lagrange polynomials.
\(^3\)Hint: all the \( b_j \) are simple fractions. Furthermore \( 70b_j \) is always an integer.
\(^4\)Since a periodic interval is used, the index \( k \) should wrap around, so that \( f_{-1} \) is equivalent to \( f_{N-1}, \text{ etc.} \).
2. A spline for exponential growth [10]. Let \( f(x) = 10^x \). We aim to construct a piecewise quadratic spline \( s(x) \) using \( N \) equally-sized intervals over the interval \([0, 1]\). Define \( h = 1/N \), and let \( s_k(x) \) be the spline over the range \([kh, (k+1)h]\) for \( k = 0, 1, \ldots, N-1 \). Each \( s_k(x) = a_kx^2 + b_kx + c_k \) is a quadratic, and hence the spline has \( 3N \) degrees of freedom in total.

(a) By any means necessary, write a program that calculates the spline coefficients according to the following constraints:

- Each \( s_k(x) \) should match the function values at both of its endpoints, so that \( s_k(0) = f(0) \) and \( s_k((k+1)h) = f((k+1)h) \). (Provides 2\( N \) constraints.)
- At each interior boundary, the spline should be differentiable, so that \( s'_k((k+1)h) = s'_{k-1}(kh) \) for \( k = 1, \ldots, N-1 \). (Provides \( N-1 \) constraints.)
- Since \( f'(x + 1) = 10f'(x) \), let \( s'_{N-1}(1) = 10s'_0(0) \). (Provides 1 constraint.)

Since there are \( 3N \) constraints for \( 3N \) degrees of freedom, there is a unique solution.

(b) For \( N = 3 \), plot the spline \( s \) and the function \( f \) on the same axes.

(c) Calculate the integral \( I[f] = \int_0^1 f(x)dx \) by hand. Write a program that exactly evaluates the integral of the spline,

\[
I[s] = \int_0^1 s(x)dx.
\]  

(4)

Make a log–log plot of \( E_h = |I[f] - I[s]| \) as a function of \( h \), using \( N = 5, 10, 15, 20, \ldots, 60 \) intervals. By fitting a function \( E_h \sim ah^b \), determine the rate of convergence.

3. Stability of a numerical scheme for advection [10]. Consider the linear advection equation

\[ u_t + cu_x = 0 \]

where \( c > 0 \), and define a discretized solution \( U^n_j \approx u(n\Delta t, j\Delta x) \). Define \( \nu = c\Delta t/\Delta x \). Using a forward Euler step for the time derivative and a one-sided second-order formula for the spatial derivative yields the numerical scheme

\[
\frac{U^{n+1}_j - U^n_j}{\Delta t} + c\frac{3U^n_j - 4U^n_{j-1} + U^n_{j-2}}{2\Delta x} = 0.
\]  

(5)

(a) By substituting in the ansatz \( U^n_j = (\lambda(k))^{n_0}e^{ijk\Delta x} \), calculate an expression for the amplification factor \( \lambda(k) \).

(b) Define \( A(k) = |\lambda(k)|^2 \). Calculate a Taylor series for \( A \) at \( k = 0 \) up to second order.\(^5\)

Using the Taylor series, explain why we consider the numerical scheme in Eq. 5 to be unstable regardless of the choice of timestep.

(c) Make a two plots of \( A(k) \) for \( \nu = 1/100 \) using two different axis ranges:

- \( 0 \leq k\Delta x \leq 2\pi \) and \( 0.91 \leq A \leq 1.01 \),
- \( 0 \leq k\Delta x \leq 0.17 \) and \( 1 - 10^{-6} \leq A \leq 1 + 10^{-6} \).

(d) Write a program\(^6\) to simulate Eq. 5 on a periodic interval \([0, 2\pi]\) using \( N = 40 \) gridpoints, and a grid spacing of \( \Delta x = 2\pi/N \). Use the initial condition \( u(0, x) = \exp(2 \sin x) \) and

\(^5\)Specifically, you should calculate that \( A(k) = a + \beta k + \gamma k^2 + O(k^3) \) for some numbers \( a, \beta, \) and \( \gamma \).

\(^6\)This program will be similar to the transp.py example from the lectures, and you may base your program on this.
\( \nu = 1/100. \) Plot the solution for \( n = 0, 1000, 2000, 4000. \) Define the root mean squared value of the solution,

\[
M(n) = \sqrt{\frac{1}{N} \sum_{j=0}^{N-1} (U_j^n)^2}.
\]

(6)

Make a plot of \( M \) over the range from \( n = 0 \) to \( n = 100000. \) You should find that \( M \) does not grow over time, indicating that the method is stable.

(e) Using the discrete Fourier transform,\(^7\) it can be shown that an arbitrary initial condition can be written as

\[
U_j^0 = \sum_{k=0}^{N-1} \alpha_k e^{ijk\Delta x}.
\]

(7)

for some constants \( \alpha_k. \) Write down an expression for the general solution \( U_j^n. \) Using this solution, show how your result in part (d) does not contradict the result in part (b).

4. Tracing out a new shape [12]. Let \( C \) be a smooth closed curve in the plane. Take a straight rod of total length \( a + b \) with endpoints labeled \( S \) and \( T. \) Mark a point \( P \) on the rod that is a distance \( a \) from \( S, \) and a distance \( b \) from \( T. \) Place the rod so that both points \( S \) & \( T \) lie on the curve. Now, move the rod around the curve so that \( S \) and \( T \) continually remain on the curve. In doing this, the point \( P \) will trace out a second curve \( D \) in the plane (Fig. 1).

Let the curve \( C \) be parametrically described by the periodic function \( x(s) = (x(s), y(s)) \) for \( s \in [0, 2\pi). \) Suppose that \( S \) is located at coordinate \( s, \) and position \( x(s). \) Then the condition stated above requires that the coordinate \( t \) of \( T \) satisfies \( \|x(s) - x(t)\|_2 = a + b \) and hence \( t \) is a root of the nonlinear function

\[
F(s, t) = (x(s) - x(t))^2 + (y(s) - y(t))^2 - (a + b)^2,
\]

(8)

where \( s \) is held at a fixed value. Write \( t = \Theta(s) \) to be the value of \( t \) corresponding to a given \( s. \)\(^8\) Then the position \( \tilde{x}(s) = (\tilde{x}(s), \tilde{y}(s)) \) of \( P \) will trace out a curve \( D \) described by

\[
\tilde{x}(s) = \frac{bx(s) + ax(\Theta(s))}{a + b}.
\]

(9)

For this question, we will make use of a curve \( C_L \) described by

\[
x(s) = r(s) \cos s, \quad y(s) = r(s) \sin s
\]

(10)

where \( r(s) = 4.2 + \cos(Ls) \) and \( L \) is an integer. Let \( a = 3/2 \) and \( b = 2/3. \) For each curve \( C_L, \) there is a corresponding curve \( D_L \) defined according to Eq. 9.

(a) Use \( L = 5. \) Calculate \( \partial F / \partial t, \) and use it to write your own Newton–Raphson method to find \( \Theta(0) \) starting from an initial guess of \( t_0 = 0.36. \) Terminate the method once \( |F| < 10^{-13}. \) Make a plot of \( \Theta(s) \) for \( s \in [0, 2\pi) \) using plot increments of \( h = \pi/400. \) \( \Theta(s) \) should be an increasing, differentiable function, and you should initialize your Newton–Raphson routine appropriately in order to find the correct root of \( F(s, t). \)

\(^7\)No knowledge of the discrete Fourier transform is required, other than the formula quoted in Eq. 7.

\(^8\)Use the convention shown in Fig. 1, where \( t \) is ahead of \( s. \) Due to periodicity, the coordinate \( s \) can be treated as equivalent to \( s + 2\pi n \) for \( n \in \mathbb{Z}. \)
(b) For the four cases of \( L = 3, 4, 5, 6 \), make graphs of \( C_L \) by calculating \( x(s) \) using increments of \( h = \pi/400 \). On each graph, plot the corresponding \( D_L \) using Eq. 9.

(c) The area enclosed by the curve \( C_L \) is given by \( A[C_L] = \int_{0}^{2\pi} f(s)ds \), where

\[
f(s) = \frac{x(s)y'(s) - y(s)x'(s)}{2}.
\]  

(11)

This integral can be accurately approximated\(^9\) by \( A[C_L] \approx h \sum_{l=0}^{N-1} f(lh) \) where \( N = 2\pi/h \). Write a program to calculate the areas enclosed by \( C_L \) for the four cases of \( L = 3, 4, 5, 6 \), reporting your results to at least six significant figures.

(d) Now calculate the area \( A[D_L] \) enclosed by \( D_L \) for \( L = 3, 4, 5, 6 \) using the same procedure as in part (c) but with the integrand

\[
\tilde{f}(s) = \frac{\tilde{x}(s)\tilde{y}'(s) - \tilde{y}(s)\tilde{x}'(s)}{2}.
\]  

(12)

In your program, \( \tilde{f}(s) \) should be evaluated exactly, and you will find that this requires evaluating \( \Theta'(s) \) exactly. To calculate this you may find it useful consider the expression

\[
\frac{d}{ds} [F(s, \Theta(s))].
\]  

(13)

(e) Calculate \( A[C_L] - A[D_L] \) for each \( L = 3, 4, 5, 6 \), reporting your results to at least six significant figures.

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\(^9\)As shown on question 1(b) of homework 3, this formula is highly accurate for evaluating integrals of smooth periodic functions.

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**Figure 1: Diagram of the curve construction procedure in question 4.** (a) A rod of length \( a + b \) is placed so its endpoints \( S \) and \( T \) are on a smooth curve \( C \); the point \( P \) on the rod is a distance \( a \) from \( S \). (b) \( S \) and \( T \) move anticlockwise around \( C \), moving in the direction of the pink arrows, causing \( P \) to trace out a new curve. (c) The curve \( D \) after the rod has moved around a complete cycle.